Math 53: Worksheet 6

October 12

1. Compute:
   (a) ∇f for $f(x, y, z) = x^2 \sin(yz)$.
   (b) Directional derivative of $g(x, y, z) = y^2 e^x - \cos(xz)$ at $(0, -1, \pi/2)$ in the direction of $\mathbf{u} = (1, 5, -4)$.
   (c) Equation of tangent plane to the surface $xy + yz + zx + 3 = e^{xyz}$ at $(-1, 2, 0)$.

2. Let $\mathbf{u} = (a, b)$ be a unit vector and let $f(x, y)$ have continuous second-order partial derivatives. Find an expression for $D_{\mathbf{u}}(D_{\mathbf{u}} f(x, y))$.

3. The temperature T in an infinite ball is inversely proportional to the distance from the center of the ball, which we take to be the origin. The temperature at the point $(2, 1, 2)$ is 60° F.
   (a) Find the rate of change of T at $(-2, 1, 0)$ in the direction toward the point $(4, 1, -1)$.
   (b) Show that at any point in the ball the direction of greatest increase in temperature is given by a vector that points towards the origin.

4. Consider a swimming pool with the temperature of the water at (x, y, z) given by $T(x, y, z)$. A fish swims through the water with position at time t given by $\mathbf{p}(t)$.
   (a) What is the directional derivative of T in the direction that the fish is traveling in at time 0?
   (b) The fish feels a temperature at any point in time. How fast does the temperature that the fish feels change, at time 0?
   (c) At $t = 1$, the fish decides it is happy with its current temperature. Describe/specify a set of directions (vectors) in which the fish should swim.
   (d) The fish changes its mind instantaneously at time $t = 1$. It goes in the direction such that the water gets colder, fastest. Give a vector pointing in this direction.

5. Let $f(x, y, z, t)$ be a smooth function and let $\nabla f = \langle f_x, f_y, f_z \rangle$ be the gradient in the *space* variables only. Let $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ be a smooth curve and $\mathbf{v}(t) = \mathbf{r}'(t)$.

(a) Show that

$$\frac{Df}{Dt} := \frac{d}{dt} f(\mathbf{r}(t), t) = \frac{\partial f}{\partial t} + \nabla f \cdot \mathbf{v}.$$

Such a derivative is commonly referred to in fluid dynamics as the material (or convective) derivative and measures the rate of change along a moving path of some physical quantity which is being transported by fluid currents.

(b) Let ρ be the density of the fluid. A fluid flow is said to be *incompressible* if

$$\frac{D\rho}{Dt} = 0.$$

Suppose further that the density depends only on the space variables (x, y, z) but not (explicitly) on t so that $\rho = \rho(x, y, z)$. An incompressible flow in this case is called *stratified*.

Show that $\nabla\rho \cdot \mathbf{v} = 0$ for stratified flow and interpret this condition.

(c) A flow is called *steady* if the density ρ and the velocity field $\mathbf{v}$ do not explicitly depend on t, i.e, $\rho = \rho(x, y, z)$ and $\mathbf{v} = \mathbf{v}(x, y, z)$. In this case, the term *streamlines* is used for the paths of the particles in the flow since they keep their shape over time.

Suppose one has a 2D stratified steady flow so that $\rho = \rho(x, y)$ and $\mathbf{v} = \mathbf{v}(x, y)$ and suppose also that the density varies only by the height y. Draw a picture of the streamlines for such a flow and explain why the term "stratified" makes sense.